The exceptional Lie algebra $E_{7(25)}$: multiplets and invariant differential operators

2009 J. Phys. A: Math. Theor. 42 285203

(http://iopscience.iop.org/1751-8121/42/28/285203)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 285203 (16pp) doi:10.1088/1751-8113/42/28/285203

The exceptional Lie algebra $E_{7(-25)}$: multiplets and invariant differential operators

V K Dobrev

Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences 72 Tsarigradsko Chaussee, 1784 Sofia, Bulgaria
and
The Abdus Salam International Centre for Theoretical Physics, PO Box 586, Strada Costiera 11, 34014 Trieste, Italy

Received 3 March 2009, in final form 6 March 2009
Published 24 June 2009
Online at stacks.iop.org/JPhysA/42/285203

Abstract
In the present paper, we continue the project of systematic construction of invariant differential operators on the example of the non-compact exceptional algebra $E_{7(-25)}$. Our choice of this particular algebra is motivated by the fact that it belongs to a narrow class of algebras, which we call 'conformal Lie algebras', which have very similar properties to the conformal algebras of n-dimensional Minkowski spacetime. This class of algebras is identified and summarized in a table. Another motivation is related to the AdS/CFT correspondence. We give the multiplets of indecomposable elementary representations, including the necessary data for all relevant invariant differential operators.

PACS numbers: 02.20.Qs, 02.20.Sv, 11.25.Hf
Mathematics Subject Classification: 17B10, 22E47, 81R05

1. Introduction

1.1. Generalities

Recently, there has been more interest in the study and applications of exceptional Lie groups; cf, e.g., [1–18]. Thus, in the development of our project [19] of systematic construction of invariant differential operators for non-compact Lie groups, we decided to give priority to some exceptional Lie groups. We start with the more interesting ones—the only two exceptional Lie groups/algebras that have highest/lowest weight representations, namely, $E_{6(-14)}$, cf [20], and $E_{7(-25)}$, which we consider in the present paper.

In fact, there are additional motivations for the choice of $E_{7(-25)}$, namely, it belongs to a narrow class of algebras, which we call 'conformal Lie algebras', which have very similar properties to the conformal algebras, $so(n,2)$, of n-dimensional Minkowski spacetime. Another motivation is related to the AdS/CFT correspondence.

Thus, we expand our motivations in the following subsection, where we also give the table of the conformal Lie algebras.

Further the paper is organized as follows. In section 2, we give the preliminaries, actually recalling and adapting facts from [19]. In section 3 we specialize to the $E_{7(-25)}$ case. In section 4, we present our results on the multiplet classification of the representations and intertwining differential operators between them. In subsection 4.1, we make a brief interpretation of our results to relate to the usual conformal algebras.

1.2. Motivation: the class of conformal Lie algebras

The group-theoretical interpretation of the AdS/CFT correspondence [21], or more general *holography*, involves two standard decompositions valid for any non-compact semi-simple Lie group G or Lie algebra $\mathcal{G}$ (also super-group/algebra): the *Iwasawa* decomposition:

$$G = KAN, \qquad \mathcal{G} = \mathcal{K} \oplus \mathcal{A} \oplus \mathcal{N}, \tag{1.1}$$

where K is the maximal compact subgroup of G, A is the Abelian simply connected subgroup of G,[1] N is a nilpotent simply connected subgroup of G preserved by the action of A (and similarly for the algebra decomposition)[2], and the *Bruhat* decomposition:

$$G = MAN\tilde{N}, \qquad \mathcal{G} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N} \oplus \tilde{\mathcal{N}}, \tag{1.2}$$

where M is a maximal subgroup of K that commutes with A, $\tilde{N}$ is a subgroup conjugate to N by the Cartan involution[3]. The Iwasawa decomposition is used to define induced representations on the bulk, which in this approach is represented by the solvable subgroup AN, while the Bruhat decomposition is used to define induced representations on the conformal boundary, i.e., on spacetime, represented by the subgroup N [21].

The application of the group-theoretical approach in [21] for the Euclidean conformal group $G = SO(n+1,1)$ was facilitated by the fact that in the group–subgroup chain $G \supset K \supset M$, the subgroups were sufficiently large: $K = SO(n+1)$, $M = SO(n)$. Thus, there was not much freedom when embedding representations, in particular, embedding the representations of $SO(n)$ into those of $SO(n+1)$.

Since the non-compact exceptional Lie algebra, $E_{7(+7)}$, was prominently used recently, cf [13], we would like to apply a similar interpretation to its holography. However, there is the problem of subgroups being not large enough. In fact, while the maximal compact subalgebra is $\mathcal{K} = su(8)$, the corresponding subalgebra $\mathcal{M}$ is null, $\mathcal{M} = \{0\}$, and the Bruhat decomposition is just $\mathcal{G} = \mathcal{A} \oplus \mathcal{N} \oplus \tilde{\mathcal{N}}$. The reason is that $E_{7(+7)}$ is maximally split; in fact, it is just the restriction to the real numbers of the complex Lie algebra E_7.

In fact, that would be a general problem in the case when the dimension r of the subalgebra $\mathcal{A}$, called *real rank* or *split rank*, is bigger than 1. But that also contains possible solutions of the problem, since when $r > 1$ the algebra under consideration has more Bruhat decompositions; in fact, the number of them is $2^r - 1$. They are written in a similar way (writing only the algebra version):

$$\mathcal{G} = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}' \oplus \tilde{\mathcal{N}}', \tag{1.3}$$

[1] Actually, $A \cong SO(1,1) \times \cdots \times SO(1,1)$, $r = \dim A$ copies.

[2] The group decomposition is global which means that each element g of G can be represented by the group multiplication of three elements from the respective subgroups: $g = kan$, $k \in K$, $a \in A$, $n \in N$. Similarly, each element, $W \in \mathcal{G}$, can be represented as the sum: $W = X \oplus Y \oplus Z$, $X \in \mathcal{K}$, $Y \in \mathcal{A}$, $Z \in \mathcal{N}$.

[3] This group decomposition is almost global, which means that the decomposition $g = man\tilde{n}$ ($m \in M, \tilde{n} \in \tilde{N}$) is valid except for a subset of G of lower dimensionality. But the algebra decomposition, $W = U \oplus Y \oplus Z \oplus \tilde{Z}$ ($U \in \mathcal{M}$, $\tilde{Z} \in \tilde{\mathcal{N}}$), is valid as above for each element $W \in \mathcal{G}$.

so that $\mathcal{M}' \supset \mathcal{M}$, $\mathcal{A}' \subset \mathcal{A}$, $\mathcal{N}' \subset \mathcal{N}$, $\tilde{\mathcal{N}}' \subset \tilde{\mathcal{N}}$. Especially useful are the so-called 'maximal' decompositions, when $\dim \mathcal{A}' = 1$, since they represent more closely the case $r = 1$, and the idea that the dimensions of the bulk (with Lie algebra $\mathcal{A}'\mathcal{N}'$) and the boundary (with Lie algebra $\mathcal{N}'$) should differ by 1.

In the case of $E_{7(+7)}$, there are several suitable Bruhat decompositions [19][4]:

$$\begin{aligned} &E_{7(+7)} = \mathcal{M}_1 \oplus \mathcal{A}_1 \oplus \mathcal{N}_1 \oplus \tilde{\mathcal{N}}_1, && \mathcal{M}_1 = so(6,6), \\ &\dim \mathcal{A}_1 = 1, && \dim \mathcal{N}_1 = \dim \tilde{\mathcal{N}}_1 = 33, \end{aligned} \tag{1.4}$$

$$\begin{aligned} &E_{7(+7)} = \mathcal{M}_2 \oplus \mathcal{A}_2 \oplus \mathcal{N}_2 \oplus \tilde{\mathcal{N}}_2, && \mathcal{M}_2 = E_{6(+6)}, \\ &\dim \mathcal{A}_2 = 1, && \dim \mathcal{N}_2 = \dim \tilde{\mathcal{N}}_2 = 27. \end{aligned} \tag{1.5}$$

Due to the presence of the subalgebra $so(6, 6)$, the first case deserves separate study. The decomposition (1.5) is mentioned, though not in our context, in [22], where it is called three-graded decomposition, and in [13], thus, it may be useful in applications to supergravity. However, instead of using the Bruhat decomposition (1.5), we shall use another non-compact real form of E_7, namely, the Lie algebra $E_{7(-25)}$.

There are several motivations to use the non-compact exceptional Lie algebra $E_{7(-25)}$. Unlike $E_{7(+7)}$ it has discrete series representations. Even more important is that it is one of two exceptional non-compact groups that have highest/lowest weight representations[5].

The groups that have highest/lowest weight representations are called Hermitian symmetric spaces [23]. The corresponding non-compact Lie algebras are

$$su(m,n), \qquad so(n,2), \qquad sp(2n,R), \qquad so^*(2n), \qquad E_{6(-14)}, \qquad E_{7(-25)}, \tag{1.6}$$

cf, e.g., [24]. The practical criterion is that in these cases the maximal compact subalgebras are of the form

$$\mathcal{K} = \mathcal{K}' \oplus so(2). \tag{1.7}$$

The most widely used of these algebras are the conformal algebras $so(n, 2)$ in n-dimensional Minkowski spacetime. In that case, there is a maximal Bruhat decomposition that has direct physical meaning:

$$\begin{aligned} &so(n,2) = \mathcal{M}_c \oplus \mathcal{A}_c \oplus \mathcal{N}_c \oplus \tilde{\mathcal{N}}_c, \\ &\mathcal{M}_c = so(n-1,1), \qquad \dim \mathcal{A}_c = 1, \qquad \dim \mathcal{N}_c = \dim \tilde{\mathcal{N}}_c = n. \end{aligned} \tag{1.8}$$

Indeed, $\mathcal{M}_c = so(n-1, 1)$ is the Lorentz algebra of n-dimensional Minkowski spacetime, the subalgebra $\mathcal{A}_c = so(1, 1)$ represents the dilatations, and the conjugated subalgebras $\mathcal{N}_c$, $\tilde{\mathcal{N}}_c$ are the algebras of translations and special conformal transformations, both being isomorphic to n-dimensional Minkowski spacetime[6].

There are other special features which are important. In particular, the complexification of the maximal compact subgroup coincides with the complexification of the first two factors of the Bruhat decomposition (1.8):

$$\mathcal{K}^{\mathbb{C}} = so(n,\mathbb{C}) \oplus so(2,\mathbb{C}) = so(n-1,1)^{\mathbb{C}} \oplus so(1,1)^{\mathbb{C}} = \mathcal{M}_c^{\mathbb{C}} \oplus \mathcal{A}_c^{\mathbb{C}}. \tag{1.9}$$

In particular, the coincidence of the complexification of the semi-simple subalgebras in (1.9), $so(n,\mathbb{C}) = so(n-1,1)^{\mathbb{C}}$, means that the sets of finite-dimensional (non-unitary) representations of $\mathcal{M}_c$ are in one-to-one correspondence with the finite-dimensional (unitary)

[4] The number of maximal Bruhat decompositions is equal to r.

[5] The other one is $E_{6(-14)}$ which we have also started to study [20].

[6] The Bruhat-decomposition interpretation of the conformal subgroups/subalgebras was done first in the Euclidean case, cf [25], then in the Minkowski case, cf [26]; for the general picture, see [27].

Table 1. Table of conformal Lie algebras.

$\mathcal{G}$	$\mathcal{K}'$	$\mathcal{M}_c$	$\dim_{\mathbb{R}} \mathcal{N}_c$
$su(n,n)$	$su(n) \oplus su(n)$	$sl(n,\mathbb{C})_{\mathbb{R}}$	n^2
$so(n,2)$ $n > 4$	$so(n)$	$so(n-1,1)$	n
$sp(n,\mathbb{R})$ $n \geqslant 2$	$su(n)$	$sl(n,\mathbb{R})$	$\frac{1}{2}(n+1)n$
$so^*(2n)$ n, even, $n \geqslant 6$	$su(n)$	$su^*(n)$	$\frac{1}{2}n(n-1)$
$E_{7(-25)}$	e_6	$E_{6(-26)}$	27

representations of $so(n)$. The latter leads to the fact that the induced representations that we consider in this paper (and which are of the type that is mostly used in physics), cf the following section, are representations of finite $\mathcal{K}$-type [23]. The role of the Abelian factors in (1.9) for the construction of highest/lowest weight representations was singled out first in [28].

It turns out that some of the algebras in (1.6) share the above-mentioned special properties of $so(n,2)$. That is why, in view of applications to physics, these algebras, together with the appropriate Bruhat decompositions should be called 'conformal Lie algebras' (resp. 'conformal Lie groups' in the group setting). We display all these algebras in table 1, where we display only the semi-simple part $\mathcal{K}'$ of $\mathcal{K}$, $sl(n,\mathbb{C})_{\mathbb{R}}$ denotes $sl(n,\mathbb{C})$ as a real Lie algebra (thus, $(sl(n,\mathbb{C})_{\mathbb{R}})^{\mathbb{C}} = sl(n,\mathbb{C}) \oplus sl(n,\mathbb{C})$), e_6 denotes the compact real form of E_6, and we have imposed restrictions to avoid coincidences or inconsistency due to well-known isomorphisms: $so(1,2) \cong sp(1,\mathbb{R}) \cong su(1,1)$, $so(2,2) \cong so(1,2) \oplus so(1,2)$, $so(3,2) \cong sp(2,\mathbb{R})$, $so(4,2) \cong su(2,2)$, $so^*(4) \cong so(3) \oplus so(2,1)$, $so^*(8) \cong so(6,2)$.

The same class was identified from different considerations in [29], where these groups/algebras were called 'conformal groups of simple Jordan algebras'. It was identified from still different considerations also in [30], where the objects of the class were called simple spacetime symmetries generalizing conformal symmetry.

Finally, we should mention that the algebra, $E_{7(-25)}$, was applied to the classification of orbits of BPS black holes in $N = 2$ Maxwell–Einstein supergravity theories [31].

With these motivations in mind, we continue with the algebra, $E_{7(-25)}$, with the following maximal Bruhat decomposition:

$$\begin{aligned} &E_{7(-25)} = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}' \oplus \tilde{\mathcal{N}}', && \mathcal{M}' = E_{6(-26)}, \\ &\dim \mathcal{A}' = 1, && \dim \mathcal{N}' = \dim \tilde{\mathcal{N}}' = 27. \end{aligned} \tag{1.10}$$

The careful reader may note that the above Bruhat decomposition is a Wick-rotation of the corresponding one for $E_{7(+7)}$, (1.5), yet there are crucial differences in their properties.

The following section contains preliminaries which are general for our programme started in [19].

2. Preliminaries

This section can be read independently from the introduction. Let G be a semi-simple non-compact Lie group, and K a maximal compact subgroup of G. Then we have an Iwasawa decomposition, $G = KAN$, where A is the Abelian simply connected vector subgroup of G, N is a nilpotent simply connected subgroup of G preserved by the action of A. Further, let M

be the centralizer of A in K. Then the subgroup, $P_0 = MAN$, is a minimal parabolic subgroup of G. A parabolic subgroup $P = M'A'N'$ is any subgroup of G (including G itself) which contains a minimal parabolic subgroup[7].

The importance of the parabolic subgroups comes from the fact that the representations induced from them generate all (admissible) irreducible representations of G [33]. For the classification of all irreducible representations it is enough to use only the so-called *cuspidal* parabolic subgroups, $P = M'A'N'$, singled out by the condition that rank $M' = \text{rank} M' \cap K$ [34, 35], so that M' has discrete series representations [36]. However, often induction from non-cuspidal parabolics is also convenient; cf [19, 37, 38].

Let ν be a (non-unitary) character of A', $\nu \in \mathcal{A}'^*$, and let μ fix an irreducible representation D^μ of M' on a vector space V_μ.

We call the induced representation, $\chi = \text{Ind}_P^G(\mu \otimes \nu \otimes 1)$, an *elementary representation* (ER) of G [25]. (These are called *generalized principal series representations* (or *limits thereof*) in [39].) Their spaces of functions are

$$\mathcal{C}_\chi = \{\mathcal{F} \in C^\infty(G, V_\mu) \mid \mathcal{F}(gman) = \mathrm{e}^{-\nu(H)} \cdot D^\mu(m^{-1})\mathcal{F}(g)\}, \qquad (2.1)$$

where $a = \exp(H) \in A'$, $H \in \mathcal{A}'$, $m \in M'$, $n \in N'$. The representation action is the *left* regular action:

$$(\mathcal{T}^\chi(g)\mathcal{F})(g') = \mathcal{F}(g^{-1}g'), \qquad g, g' \in G. \qquad (2.2)$$

For our purposes we need to restrict to *maximal* parabolic subgroups P (so that rank $A' = 1$) that may not be cuspidal. For the representations that we consider the character ν is parametrized by a real number d, called the conformal weight or energy.

Further, let μ fix a discrete series representation D^μ of M' on the Hilbert space V_μ, or the so-called limit of a discrete series representation (cf [39]). Actually, instead of the discrete series we can use the finite-dimensional (non-unitary) representation of M' with the same Casimirs.

An important ingredient in our considerations is the *highest/lowest weight representations* of $\mathcal{G}$. These can be realized as (factor modules of) Verma modules V^Λ over $\mathcal{G}^\mathbb{C}$, where $\Lambda \in (\mathcal{H}^\mathbb{C})^*$, $\mathcal{H}^\mathbb{C}$ is a Cartan subalgebra of $\mathcal{G}^\mathbb{C}$, and weight $\Lambda = \Lambda(\chi)$ is determined uniquely from χ [27]. In this setting we can consider also unitarity, which here means positivity w.r.t. the Shapovalov form in which the conjugation is the one singling out $\mathcal{G}$ from $\mathcal{G}^\mathbb{C}$.

Actually, since our ERs may be induced from finite-dimensional representations of $\mathcal{M}'$ (or their limits) the Verma modules are always reducible. Thus, it is more convenient to use *generalized Verma modules*, $\tilde{V}^\Lambda$, such that the role of the highest/lowest weight vector v_0 is taken by the (finite-dimensional) space $V_\mu v_0$. For the generalized Verma modules (GVMs) the reducibility is controlled only by the value of the conformal weight d. Relatedly, for the intertwining differential operators only the reducibility w.r.t. non-compact roots is essential.

One main ingredient of our approach is as follows. We group the (reducible) ERs with the same Casimirs in sets called *multiplets* [27, 40]. The multiplet corresponding to the fixed values of the Casimirs may be depicted as a connected graph, the vertices of which correspond to the reducible ERs and the lines between the vertices correspond to intertwining operators[8]. The explicit parametrization of the multiplets and of their ERs is important for understanding the situation.

In fact, the multiplets contain explicitly all the data necessary to construct the intertwining differential operators. Actually, the data for each intertwining differential operator consist of

[7] The number of non-conjugate parabolic subgroups is 2^r, where $r = \text{rank}\, A$; cf, e.g., [32].

[8] For simplicity, only the operators which are not compositions of other operators are depicted.

the pair (β, m), where β is a (non-compact) positive root of $\mathcal{G}^{\mathbb{C}}$, $m \in \mathbb{N}$, such that the BGG [41] Verma module reducibility condition (for highest weight modules) is fulfilled:

$$(\Lambda + \rho, \beta^{\vee}) = m, \qquad \beta^{\vee} \equiv 2\beta/(\beta, \beta). \tag{2.3}$$

When (2.3) holds then the Verma module with shifted weight $V^{\Lambda - m\beta}$ (or $\tilde{V}^{\Lambda - m\beta}$ for GVM and β non-compact) is embedded in the Verma module V^{Λ} (or $\tilde{V}^{\Lambda}$). This embedding is realized by a singular vector v_s determined by a polynomial $\mathcal{P}_{m,\beta}(\mathcal{G}^-)$ in the universal enveloping algebra $(U(\mathcal{G}_-))v_0$, $\mathcal{G}^-$ is the subalgebra of $\mathcal{G}^{\mathbb{C}}$ generated by the negative root generators [42]. More explicitly, [27], $v^s_{m,\beta} = \mathcal{P}_{m,\beta} v_0$ (or $v^s_{m,\beta} = \mathcal{P}_{m,\beta} V_{\mu} v_0$ for GVMs).[9] Then there exists [27] an intertwining differential operator,

$$\mathcal{D}_{m,\beta} : \mathcal{C}_{\chi(\Lambda)} \longrightarrow \mathcal{C}_{\chi(\Lambda - m\beta)}, \tag{2.4}$$

given explicitly by

$$\mathcal{D}_{m,\beta} = \mathcal{P}_{m,\beta}(\widehat{\mathcal{G}^-}), \tag{2.5}$$

where $\widehat{\mathcal{G}^-}$ denotes the *right* action on the functions $\mathcal{F}$; cf (2.1).

3. The non-compact Lie algebra $E_{7(-25)}$

Let $\mathcal{G} = E_{7(-25)}$. The maximal compact subgroup is $\mathcal{K} \cong e_6 \oplus so(2)$, $\dim_{\mathbb{R}} \mathcal{P} = 54$, $\dim_{\mathbb{R}} \mathcal{N} = 51$. This real form has discrete series representations and highest/lowest weight representations.

The split rank is equal to 3, while $\mathcal{M} \cong so(8)$.

The Satake diagram is [44]

$$\begin{array}{ccccccccccc} & & & & \bullet_{\alpha_2} & & & & & & \\ & & & & | & & & & & & \\ \circ & \text{———} & \bullet & \text{———} & \bullet & \text{———} & \bullet & \text{———} & \circ & \text{———} & \circ \\ \alpha_1 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_6 & & \alpha_7 \end{array} . \tag{3.1}$$

Thus, the reduced root system is presented by a Dynkin–Satake diagram looking like the C_3 Dynkin diagram:

$$\begin{array}{ccccc} \circ & \Longrightarrow & \circ & \text{———} & \circ \\ \lambda_1 & & \lambda_2 & & \lambda_3 \end{array}, \tag{3.2}$$

but the short roots have multiplicity 8 (the long ones have multiplicity 1). Going to the C_3 diagram we drop the black nodes (they give rise to $\mathcal{M}$), while $\alpha_1, \alpha_6, \alpha_7$, are mapped to $\lambda_1, \lambda_2, \lambda_3$, resp., of (3.2).

We choose a *maximal* parabolic $\mathcal{P} = \mathcal{M}'\mathcal{A}'\mathcal{N}'$ such that $\mathcal{A}' \cong so(1,1)$, while the factor $\mathcal{M}'$ has the same finite-dimensional (non-unitary) representations as the finite-dimensional (unitary) representations of the semi-simple subalgebra of $\mathcal{K}$, i.e., $\mathcal{M}' = E_{6(-6)}$; cf [19]. Thus, these induced representations are representations of finite $\mathcal{K}$-type [23]. In a related way, the number of ERs in the corresponding multiplets is equal to $|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})|/|W(\mathcal{K}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| = 56$, cf [45], where $\mathcal{H}$ is a Cartan subalgebra of both $\mathcal{G}$ and $\mathcal{K}$. Note also that $\mathcal{K}^{\mathbb{C}} \cong \mathcal{M}'^{\mathbb{C}} \oplus \mathcal{A}'^{\mathbb{C}}$. Finally, note that $\dim_{\mathbb{R}} \mathcal{N}' = 27$.

We label the signature of the ERs of $\mathcal{G}$ as follows:

$$\chi = \{n_1, \ldots, n_6; c\}, \qquad n_j \in \mathbb{N}, \qquad c = d - 9, \tag{3.3}$$

[9] For explicit expressions for singular vectors we refer to [43].

where the last entry of χ labels the characters of $\mathcal{A}'$, and the first six entries are labels of the finite-dimensional non-unitary irreducible representations (irreps) of $\mathcal{M}'$ (or of the finite-dimensional unitary irreps of e_6).

The reason to use the parameter c instead of d is that the parametrization of the ERs in the multiplets is given in a simpler way, as we shall see.

Further, we need the root system of the complex algebra E_7. With the Dynkin diagram enumerating the simple roots α_i as in (3.1), the positive roots are: first there are 21 roots forming the positive roots of $sl(7)$ with simple roots $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$, then 21 roots which are roots of the E_6 subalgebra and include the non-$sl(7)$ root α_2 :

$$\begin{gathered}
\alpha_2, \quad \alpha_2+\alpha_4, \quad \alpha_2+\alpha_4+\alpha_3, \quad \alpha_2+\alpha_4+\alpha_5, \quad \alpha_2+\alpha_4+\alpha_3+\alpha_5,\\
\alpha_2+\alpha_4+\alpha_3+\alpha_1, \quad \alpha_2+\alpha_4+\alpha_5+\alpha_6, \quad \alpha_2+\alpha_4+\alpha_3+\alpha_5+\alpha_1,\\
\alpha_2+\alpha_4+\alpha_3+\alpha_5+\alpha_6, \quad \alpha_2+\alpha_4+\alpha_3+\alpha_5+\alpha_1+\alpha_6, \quad \alpha_2+2\alpha_4+\alpha_3+\alpha_5,\\
\alpha_2+2\alpha_4+\alpha_3+\alpha_5+\alpha_1, \quad \alpha_2+2\alpha_4+\alpha_3+\alpha_5+\alpha_6, \quad \alpha_2+2\alpha_4+\alpha_3+\alpha_5+\alpha_1+\alpha_6,\\
\alpha_2+2\alpha_4+2\alpha_3+\alpha_5+\alpha_1, \quad \alpha_2+2\alpha_4+\alpha_3+2\alpha_5+\alpha_6, \quad \alpha_2+2\alpha_4+2\alpha_3+\alpha_5+\alpha_1+\alpha_6,\\
\alpha_2+2\alpha_4+\alpha_3+2\alpha_5+\alpha_1+\alpha_6, \quad \alpha_2+2\alpha_4+2\alpha_3+2\alpha_5+\alpha_1+\alpha_6,\\
\alpha_2+3\alpha_4+2\alpha_3+2\alpha_5+\alpha_1+\alpha_6, \quad 2\alpha_2+3\alpha_4+2\alpha_3+2\alpha_5+\alpha_1+\alpha_6,
\end{gathered} \tag{3.4}$$

finally, there are the following 21 roots including the non-E_6 root α_7:

$$\begin{gathered}
\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7, \quad \alpha_2+\alpha_4+\alpha_3+\alpha_5+\alpha_6+\alpha_7,\\
\alpha_2+\alpha_4+\alpha_3+\alpha_5+\alpha_1+\alpha_6+\alpha_7,\\
\alpha_2+2\alpha_4+\alpha_3+\alpha_5+\alpha_6+\alpha_7, \quad \alpha_2+2\alpha_4+\alpha_3+\alpha_5+\alpha_1+\alpha_6+\alpha_7,\\
\alpha_2+2\alpha_4+\alpha_3+2\alpha_5+\alpha_6+\alpha_7, \quad \alpha_2+2\alpha_4+2\alpha_3+\alpha_5+\alpha_1+\alpha_6+\alpha_7,\\
\alpha_2+2\alpha_4+\alpha_3+2\alpha_5+\alpha_1+\alpha_6+\alpha_7, \quad \alpha_2+2\alpha_4+2\alpha_3+2\alpha_5+\alpha_1+\alpha_6+\alpha_7,\\
\alpha_2+3\alpha_4+2\alpha_3+2\alpha_5+\alpha_1+\alpha_6+\alpha_7, \quad 2\alpha_2+3\alpha_4+2\alpha_3+2\alpha_5+\alpha_1+\alpha_6+\alpha_7,\\
\alpha_2+2\alpha_4+\alpha_3+2\alpha_5+2\alpha_6+\alpha_7,\\
\alpha_2+2\alpha_4+\alpha_3+2\alpha_5+\alpha_1+2\alpha_6+\alpha_7, \quad \alpha_2+2\alpha_4+2\alpha_3+2\alpha_5+\alpha_1+2\alpha_6+\alpha_7,\\
\alpha_2+3\alpha_4+2\alpha_3+2\alpha_5+\alpha_1+2\alpha_6+\alpha_7, \quad 2\alpha_2+3\alpha_4+2\alpha_3+2\alpha_5+\alpha_1+2\alpha_6+\alpha_7,\\
\alpha_2+3\alpha_4+2\alpha_3+3\alpha_5+\alpha_1+2\alpha_6+\alpha_7, \quad 2\alpha_2+3\alpha_4+2\alpha_3+3\alpha_5+\alpha_1+2\alpha_6+\alpha_7,\\
2\alpha_2+4\alpha_4+2\alpha_3+3\alpha_5+\alpha_1+2\alpha_6+\alpha_7,\\
2\alpha_2+4\alpha_4+3\alpha_3+3\alpha_5+\alpha_1+2\alpha_6+\alpha_7,\\
2\alpha_2+4\alpha_4+3\alpha_3+3\alpha_5+2\alpha_1+2\alpha_6+\alpha_7=\tilde{\alpha},
\end{gathered} \tag{3.5}$$

where $\tilde{\alpha}$ is the highest root of the E_7 root system.

The differential intertwining operators that give the multiplets correspond to the non-compact roots, and since we shall use the latter extensively, we introduce more compact notation for them. Namely, the nonsimple roots will be denoted in a self-explanatory way as follows:

$$\begin{aligned}
&\alpha_{ij}=\alpha_i+\alpha_{i+1}+\cdots+\alpha_j, \quad \alpha_{i,j}=\alpha_i+\alpha_j, \quad i<j,\\
&\alpha_{ij,k}=\alpha_{k,ij}=\alpha_i+\alpha_{i+1}+\cdots+\alpha_j+\alpha_k, \quad i<j,\\
&\alpha_{ij,km}=\alpha_i+\alpha_{i+1}+\cdots+\alpha_j+\alpha_k+\alpha_{k+1}+\cdots+\alpha_m, \quad i<j, \quad k<m,\\
&\alpha_{ij,km,4}=\alpha_i+\alpha_{i+1}+\cdots+\alpha_j+\alpha_k+\alpha_{k+1}+\cdots+\alpha_m+\alpha_4, \quad i<j, \quad k<m,
\end{aligned} \tag{3.6}$$

i.e., the non-compact roots will be written as

$$\alpha_7, \quad \alpha_{67}, \quad \alpha_{57}, \quad \alpha_{47}, \quad \alpha_{37}, \quad \alpha_{1,37}, \tag{3.7a}$$

$$\begin{array}{llllll} \alpha_{2,47}, & \alpha_{27}, & \alpha_{17}, & \alpha_{27,4}, & \alpha_{17,4}, & \alpha_{27,45}, \\ \alpha_{17,34}, & \alpha_{17,45}, & \alpha_{27,46}, & \alpha_{17,35}, & \alpha_{17,46}, & \alpha_{17,36}, \\ \alpha_{17,35,4}, & \alpha_{17,25,4}, & \alpha_{17,36,4}, & \alpha_{17,26,4}, & & \\ \alpha_{17,36,45}, & \alpha_{17,26,45}, & \alpha_{17,26,45,4}, & \alpha_{17,26,35,4}, & \alpha_{17,16,35,4} = \tilde{\alpha}, & \end{array} \qquad (3.7b)$$

where the first six roots in (3.7*a*) are from the $sl(7)$ subalgebra, and the 21 in (3.7*b*) are those from (3.5).

Further, we give the correspondence between the signatures χ and the highest weight Λ. The connection is through the Dynkin labels:

$$m_i \equiv \left(\Lambda + \rho, \alpha_i^\vee\right) = (\Lambda + \rho, \alpha_i), \qquad i = 1, \ldots, 7, \qquad (3.8)$$

where $\Lambda = \Lambda(\chi)$, ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$, and α_i denotes the simple roots of $\mathcal{G}^{\mathbb{C}}$. The explicit connection is

$$n_i = m_i, \qquad c = -\tfrac{1}{2}(n_{\tilde{\alpha}} + n_7) = -\tfrac{1}{2}(2n_1 + 2n_2 + 3n_3 + 4n_4 + 3n_5 + 2n_6 + 2n_7). \qquad (3.9)$$

We shall use also the so-called Harish-Chandra parameters:

$$m_\beta \equiv (\Lambda + \rho, \beta), \qquad (3.10)$$

where β is any positive root of $\mathcal{G}^{\mathbb{C}}$. These parameters are redundant, since obviously they are expressed in terms of the Dynkin labels; however, some statements are best formulated in their terms[10].

There are several types of multiplets: the main type, which contains the maximal number of ERs/GVMs, the finite-dimensional and the discrete series representations, and some reduced types of multiplets.

In the following section, we give the main type of multiplets and the main reduced type.

4. Multiplets

4.1. The main type of multiplets

The multiplets of the main type are in one-to-one correspondence with the finite-dimensional irreps of E_7, i.e., they will be labelled by the seven positive Dynkin labels $m_i \in \mathbb{N}$. As we mentioned, it turns out that each such multiplet contains 56 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm &= \left\{(m_1, m_2, m_3, m_4, m_5, m_6)^\pm; \pm\tfrac{1}{2}(m_{\tilde{\alpha}} + m_7)\right\}, \\
\chi_a^\pm &= \left\{(m_1, m_2, m_3, m_4, m_5, m_{67})^\pm; \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_7)\right\}, \\
\chi_b^\pm &= \left\{(m_1, m_2, m_3, m_4, m_{56}, m_7)^\pm; \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{67})\right\}, \\
\chi_c^\pm &= \left\{(m_1, m_2, m_3, m_{45}, m_6, m_7)^\pm; \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{57})\right\}, \\
\chi_d^\pm &= \left\{(m_1, m_{2,4}, m_{34}, m_5, m_6, m_7)^\pm; \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{47})\right\}, \\
\chi_e^\pm &= \left\{(m_1, m_4, m_{24}, m_5, m_6, m_7)^\pm; \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{2,47})\right\}, \\
\chi_{e'}^\pm &= \left\{(m_{1,3}, m_{24}, m_4, m_5, m_6, m_7)^\pm; \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{37})\right\}, \\
\chi_f^\pm &= \left\{(m_{1,3}, m_{34}, m_{2,4}, m_5, m_6, m_7)^\pm; \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{27})\right\}, \\
\chi_{f'}^\pm &= \left\{(m_3, m_{14}, m_4, m_5, m_6, m_7)^\pm; \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{1,37})\right\}, \\
\chi_g^\pm &= \left\{(m_{1,34}, m_3, m_2, m_{45}, m_6, m_7)^\pm; \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{27,4})\right\},
\end{aligned}$$

[10] Clearly, both the Dynkin and Harish-Chandra labels have their origin in the BGG reducibility condition (2.3).

$$\begin{aligned}
\chi_{g'}^{\pm} &= \big\{(m_3, m_{1,34}, m_{2,4}, m_5, m_6, m_7)^{\pm};\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{17})\big\},\\
\chi_{h}^{\pm} &= \big\{(m_{1,35}, m_3, m_2, m_4, m_{56}, m_7)^{\pm};\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{27,45})\big\},\\
\chi_{h'}^{\pm} &= \big\{(m_{34}, m_{1,3}, m_2, m_{45}, m_6, m_7)^{\pm};\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{17,4})\big\},\\
\chi_{j}^{\pm} &= \big\{(m_{1,36}, m_3, m_2, m_4, m_5, m_{67})^{\pm};\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{27,46})\big\},\\
\chi_{j'}^{\pm} &= \big\{(m_{35}, m_{1,3}, m_2, m_4, m_{56}, m_7)^{\pm};\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{17,45})\big\},\\
\chi_{j''}^{\pm} &= \big\{(m_4, m_1, m_2, m_{35}, m_6, m_7)^{\pm};\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{17,34})\big\},\\
\chi_{k}^{\pm} &= \big\{(m_{1,37}, m_3, m_2, m_4, m_5, m_6)^{\pm};\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{27,46})\big\},\\
\chi_{k'}^{\pm} &= \big\{(m_{36}, m_{1,3}, m_2, m_4, m_5, m_{67})^{\pm};\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{17,46})\big\},\\
\chi_{k''}^{\pm} &= \big\{(m_{45}, m_1, m_2, m_{34}, m_{56}, m_7)^{\pm};\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{17,35})\big\},\\
\chi_{\ell}^{\pm} &= \big\{(m_{37}, m_{1,3}, m_2, m_4, m_5, m_6)^{\pm};\, \pm\tfrac{1}{2}m_{25,34}\big\},\\
\chi_{\ell'}^{\pm} &= \big\{(m_{46}, m_1, m_2, m_{34}, m_5, m_{67})^{\pm};\, \pm\tfrac{1}{2}m_{2,45,4}\big\},\\
\chi_{\ell''}^{\pm} &= \big\{(m_5, m_1, m_{2,4}, m_3, m_{46}, m_7)^{\pm};\, \pm\tfrac{1}{2}m_{2,56}\big\},\\
\chi_{m}^{\pm} &= \big\{(m_{47}, m_1, m_2, m_{34}, m_5, m_6)^{\pm};\, \pm\tfrac{1}{2}m_{2,45,4}\big\},\\
\chi_{m'}^{\pm} &= \big\{(m_{56}, m_1, m_{2,4}, m_3, m_{45}, m_{67})^{\pm};\, \pm\tfrac{1}{2}m_{2,5}\big\},\\
\chi_{m''}^{\pm} &= \big\{(m_5, m_1, m_4, m_3, m_{2,46}, m_7)^{\pm};\, \pm\tfrac{1}{2}(m_{56} - m_2)\big\},\\
\chi_{n}^{\pm} &= \big\{(m_{57}, m_1, m_{2,4}, m_3, m_{45}, m_6)^{\pm};\, \pm\tfrac{1}{2}m_{2,5}\big\},\\
\chi_{n'}^{\pm} &= \big\{(m_6, m_1, m_{2,45}, m_3, m_4, m_{57})^{\pm};\, \pm\tfrac{1}{2}(m_2 - m_5)\big\},\\
\chi_{n''}^{\pm} &= \big\{(m_{56}, m_1, m_4, m_3, m_{2,45}, m_{67})^{\pm};\, \pm\tfrac{1}{2}(m_5 - m_2)\big\},
\end{aligned} \tag{4.1}$$

where we have used for the numbers $m_\beta = (\Lambda(\chi) + \rho, \beta)$, the same compact notation as in (3.6) for the roots β, and the notation $(\cdots)^{\pm}$ employs the natural conjugation of the subalgebra E_6, more precisely:

$$\begin{aligned}
&(n_1, n_2, n_3, n_4, n_5, n_6)^{-} = (n_1, n_2, n_3, n_4, n_5, n_6),\\
&(n_1, n_2, n_3, n_4, n_5, n_6)^{+} = (n_1, n_2, n_3, n_4, n_5, n_6)^{E_6} \doteq (n_6, n_2, n_5, n_4, n_3, n_1).
\end{aligned} \tag{4.2}$$

Note that in (4.1) the last entries with sign plus (resp. minus) are positive (resp. negative), except in the cases $\chi_m^{\pm}, \chi_n^{\pm}, \chi_{n'}^{\pm}$.

The ERs in the multiplet are related by intertwining integral and differential operators. The integral operators were introduced by Knapp and Stein [46]. In fact, these operators are defined for any ER, not only for the reducible ones, the general action being

$$\begin{aligned}
&G_{\rm KS} : \mathcal{C}_\chi \longrightarrow \mathcal{C}_{\chi'},\\
&\chi = \{n_1, n_2, n_3, n_4, n_5, n_6; c\},\\
&\chi' = \{(n_1, n_2, n_3, n_4, n_5, n_6)^{E_6}; -c\} = \{n_6, n_2, n_5, n_4, n_3, n_1; -c\}.
\end{aligned} \tag{4.3}$$

Obviously, the pairs in (4.1) are related by Knapp–Stein integral operators, i.e.,

$$G_{\rm KS} : \mathcal{C}_{\chi^{\mp}} \longrightarrow \mathcal{C}_{\chi^{\pm}}. \tag{4.4}$$

The action on the signatures is also called restricted Weyl reflection, since it represents the nontrivial element of the two-element restricted Weyl group which arises canonically with every maximal parabolic subalgebra[11].

Matters are arranged so that in every multiplet only the ER with signature χ_0^- contains a finite-dimensional non-unitary subrepresentation in a finite-dimensional subspace $\mathcal{E}$.

[11] Generically, the Knapp–Stein operators can be normalized so that indeed $G_{\rm KS} \circ G_{\rm KS} = {\rm Id}_{\mathcal{C}_\chi}$. However, this usually fails exactly for the reducible ERs that form the multiplets; cf, e.g., [25].

The latter corresponds to the finite-dimensional irrep of E_7 with signature $\{m_1, \ldots, m_7\}$. The subspace $\mathcal{E}$ is annihilated by the operator G^+, and is the image of the operator G^-. The subspace $\mathcal{E}$ is also annihilated by the intertwining differential operator acting from χ_0^- to χ_b^- (more about this operator below). When all $m_i = 1$ then $\dim \mathcal{E} = 1$, and in that case $\mathcal{E}$ is also the trivial one-dimensional UIR of the whole algebra $E_{7(-25)}$. Furthermore, in that case the conformal weight is zero: $d = 9 + c = 9 - \frac{1}{2}(m_{\tilde{\alpha}} + m_7)_{|m_i=1} = 0$.

Analogously, in every multiplet only the ER with signature χ_0^+ contains the holomorphic discrete series representation. This is guaranteed by the criterion [36] that for such an ER all Harish-Chandra parameters for non-compact roots must be negative, i.e., in our situation, $n_\alpha < 0$, for α from (3.7). (That this holds for our χ_0^+ can easily be checked using the signatures (4.1).)

In fact, the Harish-Chandra parameters are reflected in the division of the ERs into χ^- and χ^+: for the χ^- modules less than half of the 27 non-compact Harish-Chandra parameters are negative (none for χ_0^-, 13 for χ_n^-, $\chi_{n'}^-$, $\chi_{n''}^-$), while for the χ^+ modules more than half of the non-compact 27 Harish-Chandra parameters are negative (27 for χ_0^+, 14 for χ_n^+, $\chi_{n'}^+$, $\chi_{n''}^+$). In fact, as in the parenthesized examples, it is true that the sum of the number of negative Harish-Chandra parameters for any pair $\chi^\pm$ is equal to 27.

Note that the ER χ_0^+ also contains the conjugate anti-holomorphic discrete series. The direct sum of the holomorphic and the anti-holomorphic representation is realized in an invariant subspace $\mathcal{D}$ of the ER χ_0^+. That subspace is annihilated by the operator G^-, and is the image of the operator G^+.

Note that the corresponding lowest weight GVM is infinitesimally equivalent only to the holomorphic discrete series, while the conjugate highest weight GVM is infinitesimally equivalent to the anti-holomorphic discrete series. The conformal weight of the ER χ_0^+ has the restriction $d = 9 + c = 9 + \frac{1}{2}(m_{\tilde{\alpha}} + m_7) \geqslant 18$.

The intertwining differential operators correspond to non-compact positive roots of the root system of E_7, cf [27], i.e., in the current context, the roots given in (3.7).

The multiplets are given explicitly in figure 1, where we use the notation $\Lambda^\pm = \Lambda(\chi^\pm)$. Each intertwining differential operator is represented by an arrow accompanied by a symbol $i_{j\ldots k}$ encoding the root $\beta_{j\ldots k}$ and the number $m_{\beta_{j\ldots k}}$ which is involved in the BGG criterion. This notation is used to save space, but it can be used due to the fact that only intertwining differential operators which are non-composite are displayed, and that the data β, m_β which are involved in the embedding $V^\Lambda \longrightarrow V^{\Lambda - m_\beta, \beta}$ turn out to involve only the m_i corresponding to simple roots, i.e., for each β, m_β there exists $i = i(\beta, m_\beta, \Lambda) \in \{1, \ldots, 7\}$, such that $m_\beta = m_i$. Hence the data $\beta_{j\ldots k}, m_{\beta_{j\ldots k}}$ are represented by $i_{j\ldots k}$ on the arrows.

The pairs $\Lambda^\pm$ are symmetric w.r.t. to the bullet in the middle of the figure, and the dashed line separates the Λ^- modules from the Λ^+ modules.

Interpretation: since the relation to the usual conformal algebras in n-dimensional Minkowski spacetime is one of our main motivations to study $E_{7(-25)}$, we would like to mention briefly some analogies, using an exposition that is written in the same context, cf [38], though the results are contained in much older work [25, 26, 47, 48]; see also [27]. If we take the most basic example when the inducing E_6-representation in the ERs $\chi_0^\pm$ is the trivial one: $(m_1, m_2, m_3, m_4, m_5, m_6) = (1, 1, 1, 1, 1, 1)$, then the conformal fields represented by the ERs $\chi_0^\pm$ are scalar, while those represented by the ERs $\chi_a^\pm$ are 27-dimensional vectors. There are invariant differential operators depicted in figure 1:

$$\mathcal{D}_{m_7,\alpha_7} : \mathcal{C}_{\chi_0^-} \longrightarrow \mathcal{C}_{\chi_a^-}, \tag{4.5a}$$

$$\mathcal{D}_{m_7,\alpha_{17,16,35,4}} : \mathcal{C}_{\chi_a^+} \longrightarrow \mathcal{C}_{\chi_0^+}. \tag{4.5b}$$

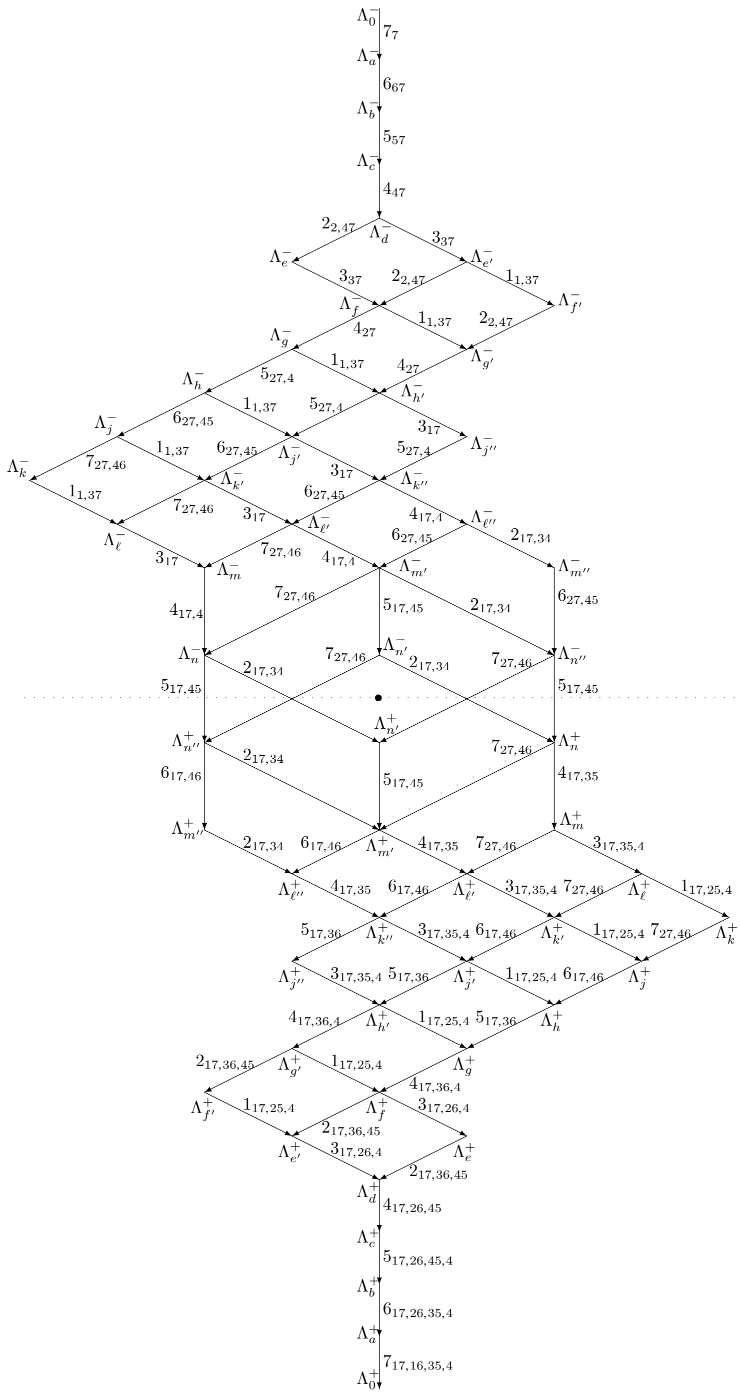

Figure 1. Main type.

Both are equations of order m_7. When the last free parameter $m_7 = 1$ then the ER χ_a^- is the analogue of the vector potential A_ν, while the ER χ_a^+ is the analogue of the current J_ν. Then the equations in (4.5) are linear and can be written as

$$\partial_\nu \phi = A_\nu, \qquad \phi \in \mathcal{C}_{\chi_0^-}, \qquad A \in \mathcal{C}_{\chi_a^-}, \tag{4.6a}$$

$$\sum_{\nu=1}^{27} \partial^\nu J_\nu = \Phi, \qquad \Phi \in \mathcal{C}_{\chi_0^+}, \qquad J \in \mathcal{C}_{\chi_a^+}. \tag{4.6b}$$

When the parameter $m_7 > 1$, then the analogues of (4.5) are also treated in the older references cited above (for instance, (4.5b) would be an equation of partial conservation). In all cases, we stress that these are *invariant* differential equations, on- and off-shell. Naturally, this is only a glimpse at the analogies with the usual conformal case, much more will be said elsewhere, [49]. $\diamond$

In the following subsection, we shall consider the main type of reduced multiplets.

4.2. The main type of reduced multiplet

The multiplets of reduced type R7 contain 42 ERs/GVMs and may be obtained formally from the main type by setting $m_7 = 0$. Their signatures are given explicitly by

$$\begin{aligned}
\chi_0^\pm &= \left\{(m_1, m_2, m_3, m_4, m_5, m_6)^\pm; \pm\tfrac{1}{2} m_{\tilde\alpha}\right\},\\
\chi_b^\pm &= \left\{(m_1, m_2, m_3, m_4, m_{56}, 0)^\pm; \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_6)\right\},\\
\chi_c^\pm &= \left\{(m_1, m_2, m_3, m_{45}, m_6, 0)^\pm; \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{56})\right\},\\
\chi_d^\pm &= \left\{(m_1, m_{2,4}, m_{34}, m_5, m_6, 0)^\pm; \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{46})\right\},\\
\chi_e^\pm &= \left\{(m_1, m_4, m_{24}, m_5, m_6, 0)^\pm; \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{2,46})\right\},\\
\chi_{e'}^\pm &= \left\{(m_{1,3}, m_{24}, m_4, m_5, m_6, 0)^\pm; \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{36})\right\},\\
\chi_f^\pm &= \left\{(m_{1,3}, m_{34}, m_{2,4}, m_5, m_6, 0)^\pm; \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{26})\right\},\\
\chi_{f'}^\pm &= \left\{(m_3, m_{14}, m_4, m_5, m_6, 0)^\pm; \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{1,36})\right\},\\
\chi_g^\pm &= \left\{(m_{1,34}, m_3, m_2, m_{45}, m_6, 0)^\pm; \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{26,4})\right\},\\
\chi_{g'}^\pm &= \left\{(m_3, m_{1,34}, m_{2,4}, m_5, m_6, 0)^\pm; \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{16})\right\},\\
\chi_h^\pm &= \left\{(m_{1,35}, m_3, m_2, m_4, m_{56}, 0)^\pm; \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{26,45})\right\},\\
\chi_{h'}^\pm &= \left\{(m_{34}, m_{1,3}, m_2, m_{45}, m_6, 0)^\pm; \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{16,4})\right\},\\
\chi_j^\pm &= \left\{(m_{1,36}, m_3, m_2, m_4, m_5, m_6)^\pm; \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{26,46})\right\},\\
\chi_{j'}^\pm &= \left\{(m_{35}, m_{1,3}, m_2, m_4, m_{56}, 0)^\pm; \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{16,45})\right\},\\
\chi_{j''}^\pm &= \left\{(m_4, m_1, m_2, m_{35}, m_6, 0)^\pm; \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{16,34})\right\},\\
\chi_{k''}^\pm &= \left\{(m_{45}, m_1, m_2, m_{34}, m_{56}, 0)^\pm; \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{16,35})\right\},\\
\chi_\ell^\pm &= \left\{(m_{36}, m_{1,3}, m_2, m_4, m_5, m_6)^\pm; \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{16,46})\right\},\\
\chi_m^\pm &= \left\{(m_{46}, m_1, m_2, m_{34}, m_5, m_6)^\pm; \pm\tfrac{1}{2} m_{2,45,4}\right\},\\
\chi_{\ell''}^\pm &= \left\{(m_5, m_1, m_{2,4}, m_3, m_{46}, 0)^\pm; \pm\tfrac{1}{2} m_{2,56}\right\},\\
\chi_{m''}^\pm &= \left\{(m_5, m_1, m_4, m_3, m_{2,46}, 0)^\pm; \pm\tfrac{1}{2}(m_{56} - m_2)\right\},\\
\chi_n^\pm &= \left\{(m_{56}, m_1, m_{2,4}, m_3, m_{45}, m_6)^\pm; \pm\tfrac{1}{2} m_{2,5}\right\}\\
\chi_{n''}^\pm &= \left\{(m_4, m_3, m_{2,45}, m_1, m_6, m_{56})^\pm; \pm\tfrac{1}{2}(m_5 - m_2)\right\}.
\end{aligned} \tag{4.7}$$

Here the ER χ_0^+ contains limits of the (anti)holomorphic discrete series representations. This is guaranteed by the fact that for this ER all Harish-Chandra parameters for non-compact

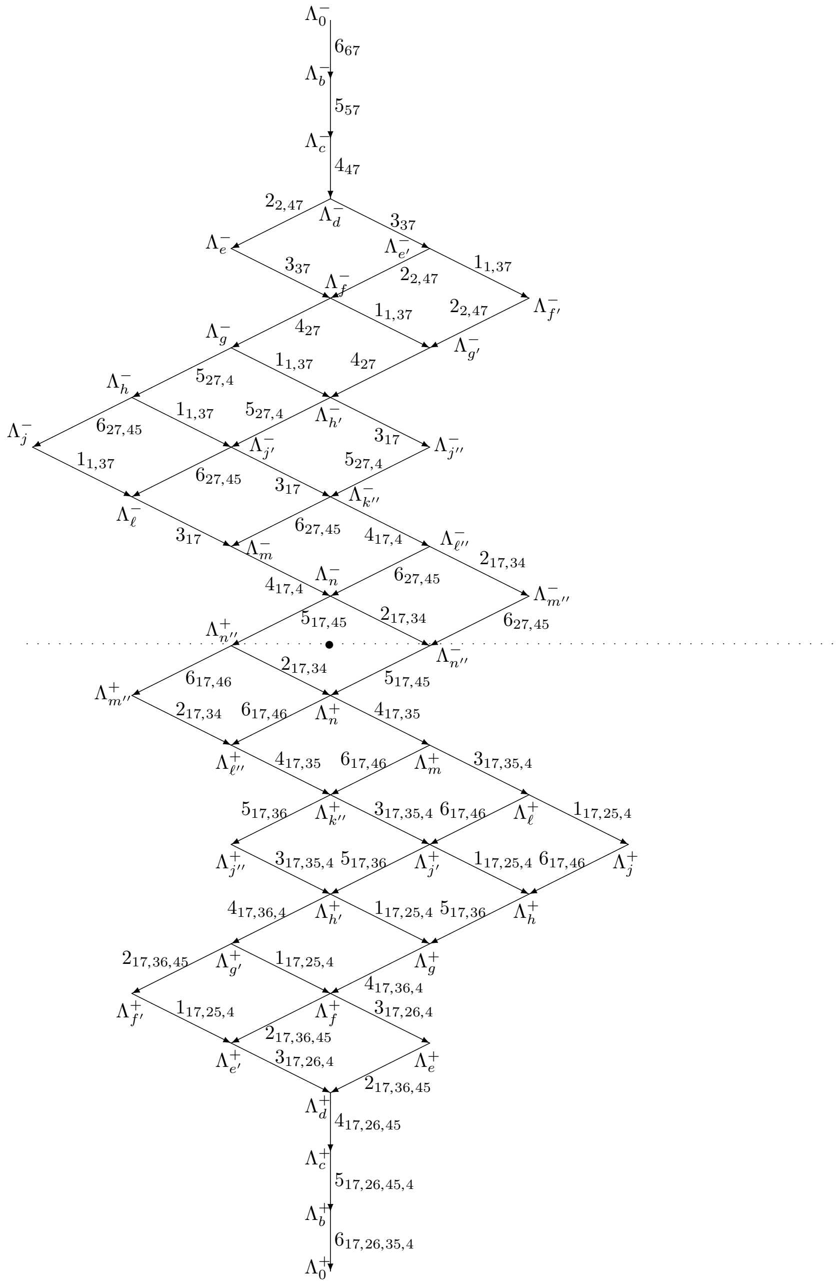

Figure 2. Reduced type R7.

roots are non-positive, i.e., $n_\alpha \leqslant 0$, for α from (3.7). The conformal weight has the restriction $d = 9 + c = 9 + \frac{1}{2}m_{\tilde{\alpha}} \geqslant 17$.

There are other limiting cases, where there are zero entries for the first six n_i values. In these cases, the induction procedure would not use finite-dimensional irreps of the E_6 subgroup. The corresponding ERs would not have direct physical meaning; however, the fact that they are together with the physically meaningful ERs has important bearing on the structure of the latter.

Altogether, the analysis of the Harish-Chandra parameters reveals the following. For any ER there is exactly one Harish-Chandra parameter (counting all, not only the non-compact) that is zero. The compact ones are seen in the list above. The non-compact are as follows:

$$\begin{aligned} &\chi_0^- : n_7 = 0, \quad \chi_0^+ : n_{\tilde\alpha} = 0, \\ &\chi_j^\pm, \quad \chi_\ell^\pm, \quad \chi_m^\pm, \quad \chi_n^\pm, \quad \chi_{n''}^\pm, : n_{27,46} = 0. \end{aligned} \tag{4.8}$$

As in the main type, for the χ^- modules less than half of the 27 non-compact Harish-Chandra parameters are negative (none for χ_0^-, 13 for $\chi_{n''}^-$), while for the χ^+ modules—except $\chi_{n''}^+$—more than half of the non-compact 27 Harish-Chandra parameters are negative (26 for χ_0^+, 14 for χ_n^+). In fact, it is true that for any pair $\chi^\pm$ the sum of the number of negative Harish-Chandra parameters is equal to 26.

These multiplets are depicted in figure 2. The Weyl-conjugated pairs $\Lambda^\pm$ are symmetric w.r.t. to the bullet in the middle of the figure, and the dashed line separates the Λ^- modules from the Λ^+ modules. The fact that the pair, $\chi_{n''}^-$, $\chi_{n''}^+$, sits on the dashed line signifies the fact that for these two ERs the number of negative non-compact Harish-Chandra parameters equals the number of positive non-compact Harish-Chandra parameters, and that equals 13. Note also that the ten ERs for which $n_{27,46} = 0$ holds, cf (4.8), are situated on two conjugated lines.

There are many other types of reduced multiplets, and their study may be done as in the case of $E_{6(-14)}$ in [20], but for $E_{7(-25)}$ it will need much more space, so we leave it for a future publication.

5. Outlook

In the present paper, we continued the programme outlined in [19] on the example of the non-compact group $E_{7(-25)}$. Similar explicit descriptions are planned for the other non-compact groups, in particular those with highest/lowest weight representations. We also plan to extend these considerations to the supersymmetric cases and also to the quantum group setting. Such considerations are expected to be very useful for applications to string theory and integrable models; cf, e.g., [50].

In our further plans it will be very useful that (as in [19]) we follow a procedure in representation theory in which intertwining differential operators appear canonically [27] and in which the procedure has been generalized to the supersymmetry setting [51, 52] and to quantum groups [53]. (For more references, cf [19].)

Acknowledgments

The author would like to thank for hospitality the Abdus Salam International Centre for Theoretical Physics, where part of the work was done. The author was supported in part by the European RTN network *'Forces-Universe'* (contract no *MRTN-CT-2004-005104*), by Bulgarian NSF grant *DO 02-257* and by the Alexander von Humboldt Foundation in the framework of the Clausthal–Leipzig–Sofia Cooperation.

References

[1] Kang J and Langacker P 2005 *Phys. Rev.* D **71** 035014 (arXiv:hep-ph/0412190)
[2] Gannon T and Vasudevan M 2005 *J. High Energy Phys.* JHEP07(2005)035 (arXiv:hep-th/0504006)
[3] Ohwashi Y 2006 *Prog. Theor. Phys.* **115** 625–59 (arXiv:hep-th/0510252)

[4] Tuite M P 2006 arXiv:math/0610322
Tuite M P 2008 arXiv:0811.4523 [math.QA]
[5] Keller C A and Rossi S 2007 *J. High Energy Phys.* JHEP03(2007)038 (arXiv:hep-th/0610175)
[6] Hung P Q and Mosconi Paola 2006 E_6 unification in a model of dark energy and dark matter arXiv:hep-ph/0611001
[7] Duff M J and Ferrara S 2007 *Phys. Rev.* D **76** 124023 (arXiv:0704.0507v2 [hep-th])
[8] Faraggi A E, Kounnas C and Rizos J 2007 *Nucl. Phys.* B **774** 208–31 (arXiv:hep-th/0611251)
[9] King S F, Moretti S and Nevzorov R 2007 *Phys. Lett.* B **650** 57–64 (arXiv:hep-ph/0701064)
[10] Henningson M and Wyllard N 2007 *J. High Energy Phys.* JHEP07(2007)084 (arXiv:0706.2803 [hep-th])
[11] Gurrieri S, Lukas A and Micu A 2007 *J. High Energy Phys.* JHEP12(2007)081 (arXiv:0709.1932)
[12] Bernardoni F, Cacciatori S L, Cerchiai Bianca L and Scotti A 2008 *J. Math. Phys.* **49** 012107 (arXiv:0710.0356)
[13] Kallosh R and Soroush M 2008 *Nucl. Phys.* B **801** 25–44 (arXiv:0802.4106)
Kallosh R and Kugo T 2008 arXiv:0811.3414
[14] Nunez J F, Fuertes W G and Perelomov A M 2008 Quantum trigonometric Calogero–Sutherland model and irreducible characters for the exceptional algebra E(8) arXiv:0806.1011 [math-ph]
[15] Sati H 2008 OP**2 bundles in M-theory arXiv:0807.4899
[16] Mizoguchi Sh 2008 Localized modes in type II and heterotic singular Calabi–Yau conformal field theories arXiv:0808.2857
[17] Howl R and King S F 2008 *J. High Energy Phys.* JHEP01(2008)060
[18] Le Diffon A and Samtleben H 2008 Supergravities without an action: gauging the trombone arXiv:0809.5180
[19] Dobrev V K 2008 *Rev. Math. Phys.* **20** 407–49 (arXiv:hep-th/0702152) (*ICTP Trieste Preprint* IC/2007/015)
[20] Dobrev V K 2008 Invariant differential operators for non-compact Lie groups: the $E_{6(-14)}$ case arXiv:0812.2655 [math-ph] (Invited talks at V School in Modern Mathematical Physics, Belgrade, Serbia, 6-17.7.2008 (to appear in the Proceedings))
[21] Dobrev V K 1999 Intertwining operator realization of the AdS/CFT correspondence *Nucl. Phys.* B **553** 559–82 (arXiv:hep-th/9812194)
[22] Gunaydin M, Koepsell K and Nicolai H 2001 *Commun. Math. Phys.* **221** 57–76 (arXiv:hep-th/0008063)
[23] Harish-Chandra 1955 Representations of semisimple Lie groups: IV *Am. J. Math.* **77** 743–77
Harish-Chandra 1956 Representations of semisimple Lie groups: V *Am. J. Math.* **78** 1–41
[24] Helgason S 1978 *Differential Geometry, Lie Groups and Symmetric Spaces* (New York: Academic)
[25] Dobrev V K, Mack G, Petkova V B, Petrova S G and Todorov I T 1977 *Harmonic Analysis on the n-Dimensional Lorentz Group and Its Applications to Conformal Quantum Field Theory (Lecture Notes in Physics* vol 63*)* (Berlin: Springer)
Dobrev V K and Petkova V B 1978 *Rep. Math. Phys.* **13** 233–77
[26] Dobrev V K 1985 *J. Math. Phys.* **26** 235–51 (first as *ICTP Trieste Preprint* IC/1983/36)
[27] Dobrev V K 1988 *Rep. Math. Phys.* **25** 159–81 (first as *ICTP Trieste Preprint* IC/1986/393)
[28] Mack G 1977 *Commun. Math. Phys.* **55** 1–28
[29] Gunaydin M 1993 *Mod. Phys. Lett.* A **8** 1407–16
[30] Mack G and de Riese M 2007 *J. Math. Phys.* **48** 052304 (arXiv:hep-th/0410277v2)
[31] Ferrara S and Gunaydin M 1998 *Int. J. Mod. Phys.* A **13** 2075–88
[32] Warner G 1972 *Harmonic Analysis on Semi-Simple Lie Groups I* (Berlin: Springer)
[33] Langlands R P 1988 *On the Classification of Irreducible Representations of Real Algebraic Groups (Math. Surveys and Monographs* vol 31*)* (Providence, RI: American Mathematical Society) (first as *IAS Princeton Preprint* (1973))
[34] Zhelobenko D P 1974 *Harmonic Analysis on Semisimple Complex Lie Groups* (Moscow: Nauka) (in Russian)
[35] Knapp A W and Zuckerman G J 1977 *Classification Theorems for Representations of Semisimple Groups (Lecture Notes in Math.* vol 587*)* (Berlin: Springer) pp 138–59
Knapp A W and Zuckerman G J 1982 Classification of irreducible tempered representations of semisimple groups *Ann. Math.* **116** 389–501
[36] Harish-Chandra 1966 Discrete series for semisimple Lie groups: II *Ann. Math.* **116** 1–111
[37] Enright T, Howe R and Wallach W 1983 A classification of unitary highest weight modules *Representations of Reductive Groups* ed P Trombi (Boston: Birkhäuser) pp 97–143
[38] Dobrev V K 2008 *J. Phys. A: Math. Theor.* **41** 425206 (arXiv:0712.4375 [hep-th])
[39] Knapp A W 1986 *Representation Theory of Semisimple Groups (An Overview Based on Examples)* (Princeton, NJ: Princeton University Press)
[40] Dobrev V K 1985 *Lett. Math. Phys.* **9** 205–11
[41] Bernstein I N, Gel'fand I M and Gel'fand S I 1971 *Funkt. Anal. Prilozh.* **5** 1–9
Bernstein I N, Gel'fand I M and Gel'fand S I 1971 *Funct. Anal. Appl.* **5** 1–8 (Engl. Transl.)

[42] Dixmier J 1977 *Enveloping Algebras* (New York: North-Holland)
[43] Dobrev V K 1991 *Lett. Math. Phys.* **22** 251–66
[44] Satake I 1960 *Ann. Math.* **71** 77–110
[45] Bourbaki N 1968 *Groupes at algèbres de Lie* ch 4, 5, 6 (Paris: Hermann)
[46] Knapp A W and Stein E M 1971 Intertwining operators for semisimple groups *Ann. Math.* **93** 489–578
Knapp A W and Stein E M 1980 Intertwining operators for semisimple groups *Inv. Math.* **60** 9–84
[47] Todorov I T, Minchev M C and Petkova V B 1978 *Conformal Invariance in QFT* (Pisa: Scuola Normale Superiore)
[48] Petkova V B and Sotkov G M 1984 *Lett. Math. Phys.* **8** 217–26
Petkova V B and Sotkov G M 1985 *Lett. Math. Phys.* **9** 83 (erratum)
[49] Dobrev V K in preparation
[50] Witten E 2007 Conformal field theory in four and six dimensions arXiv:0712.0157
[51] Dobrev V K and Petkova V B 1985 *Phys. Lett.* B **162** 127–32
Dobrev V K and Petkova V B 1985 *Lett. Math. Phys.* **9** 287–98
Dobrev V K and Petkova V B 1987 *Fortsch. Phys.* **35** 537–72
[52] Dobrev V K 1987 *Phys. Lett.* B **186** 43–51
Dobrev V K 2002 *J. Phys. A: Math. Gen.* **35** 7079–100 (arXiv:hep-th/0201076)
Dobrev V K 2007 *Phys. Part. Nucl.* **38** 564–609 (arXiv:hep-th/0406154)
Dobrev V K and Ganchev A Ch 1988 *Mod. Phys. Lett.* A **3** 127–37
[53] Dobrev V K 1992 *J. Math. Phys.* **33** 3419–30
Dobrev V K 1993 *J. Phys. A: Math. Gen.* **26** 1317–34 (first as *Göttingen University Preprint* (July 1991)
Dobrev V K 1994 *J. Phys. A: Math. Gen.* **27** 4841–57
Dobrev V K 1994 *J. Phys. A: Math. Gen.* **27** 6633–34 (arXiv:hep-th/9405150) (erratum)
Dobrev V K 1994 *Phys. Lett.* B **341** 133–8
Dobrev V K 1995 *Phys. Lett.* B **346** 427 (erratum)
Dobrev V K and Moylan P J 1993 *Phys. Lett.* B **315** 292–8